Recap

$$f_{Y_1|Y_2}(y_1|y_2) = \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_2}(y_2)}$$

$$E[g(Y_1, Y_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(y_1, y_2) f(y_1, y_2)\, dy_1\, dy_2$$

$$E[Y_1] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y_1 f(y_1, y_2)\, dy_2\, dy_1$$

$$= \int_{-\infty}^{\infty} y_1 f(y_1)\, dy_1$$

Conditional expected values

$$E[g(Y_1, Y_2) | Y_2 = y_2] = \int_{-\infty}^{\infty} g(y_1, y_2) \times f_{Y_1|Y_2}(y_1|y_2)\, dy_1$$

$$E[Y_1 | Y_2 = y_2] = g^*(y_2)$$

$P[g^*(y_2) = a]$ depends on y_2

$$g^*(Y_2) = E[Y_1 | Y_2]$$

$$E[g^*(Y_2)] = E[E[Y_1|Y_2]]$$

Thm

$$E_{f_{Y_2}(y_2)}[E_{f_{Y_1|Y_2}(y_1|y_2)}(Y_1|Y_2)] = E[Y_1]$$ Remember

start with

$$E[Y_1] = \iint y_1 f_{Y_1,Y_2}(y_1, y_2)\, dy_1\, dy_2$$

$$f_{Y_1|Y_2}(y_1|y_2) = \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_2}(y_2)}$$

$$E[Y_1] = \iint y_1 f_{Y_1|Y_2}(y_1|y_2) f_{Y_2}(y_2)\, dy_1\, dy_2$$

$$= \int f_{Y_2}(y_2) \underbrace{\int y_1 f_{Y_1|Y_2}(y_1|y_2)\, dy_1}_{E[Y_1|Y_2 = y_2]}\, dy_2$$

$$= \int f_{Y_2}(y_2) \times E[Y_1 | Y_2 = y_2]\, dy_2$$

$$= E[\, E[Y_1 | Y_2 = y_2]\,]$$

Thm

$$var(Y_1) = E[var(Y_1|Y_2)] + var(E[Y_1|Y_2])$$

$$= E[Y_1^2] - E[Y_1]^2$$

$$= E[E[Y_1^2|Y_2]] - E[E[Y_1|Y_2]]^2 + E[E[Y_1|Y_2]^2] - E[E[Y_1|Y_2]^2]$$

$$= E[\, E[Y_1^2|Y_2] - E[Y_1|Y_2]^2\,] + E[\, E[[Y_1|Y_2]^2] - E[E[Y_1|Y_2]]^2$$

$$= E[var(Y_1|Y_2)] + var(E[Y_1|Y_2])$$